2016 Preliminary Round Solutions

1. Jeffery the Jangaroo is looking for his favorite stuffed Jiraffe. He walks 10 feet west, but doesn't find it. He is disappointed, but Jeffery is a resilient Jangaroo and continues looking. He walks 100 feet east but still doesn't find it. Jeffery walks back to where he began, sits down, and cries. How far did Jeffery walk in total?

   Solution. We will directly count how far he walks. He first walks 10 feet west. Then he walks 100 feet east. At this point, we can see that he is 90 feet east from where he began. So he has to walk 90 feet back to where he began. Therefore the answer is $10 + 100 + 90 = 200$.

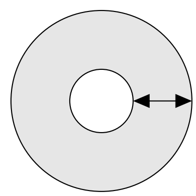
2. Bored of his circular kimchi pie, Daniel instead cooks a square kimchi pie. The square's perimeter is twice its area. Find the area of his kimchi pie.

   Solution. Let x be the length of the side of the square. Then since the perimeter is twice the area, $4x = 2x^2$. So $x = 2$. So the area of the square is $2 \cdot 2 = 4$.

3. A triangle has integer side lengths $20, 16$, and x. Find the largest possible value of x.

   Solution. By triangle inequality, $20 + 16 = 36 > x$. Therefore, maximum integral value x can be is 35.

4. Kenny the Kookie Monster likes to eat 2-dimensional donut cookies. A donut cookie is composed of the shaded region in between 2 concentric circles. The width (indicated by the arrow) of the donut is 2. If the donut cookie has area 12π, find the radius of the larger circle.

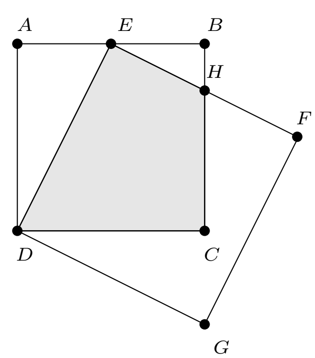


   Solution. Let the inner radius be r. Then we have that $((r + 2)^2 - r^2)\pi = 12\pi$. So $4x + 4 = 12$. Therefore the radius of the inner circle is 2, so the radius of the outer circle is 4.

5. In the diagram below, let $ABCD$ be a square with side length 1. Let E be the midpoint of AB. Square $DEFG$ is constructed with side length DE. Find the area of the shaded region.

   Solution. We notice that we can find the area of the shaded region by subtracting the areas of triangles $\triangle ADE$ and $\triangle EBH$ from the area of $ABCD$. We have that $AE = EB = 1/2$ Notice that $\triangle DAE \sim \triangle EBH$, so $BH = 1/4$. It follows that the shaded region has area

$$1 - \left(\frac{1}{2} \cdot 1 \cdot \frac{1}{2}\right) - \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4}\right) = \frac{11}{16}.$$



6. A circle in the coordinate plane passes through the points $(0, 12), (0, -20), (8, 0)$, and $(k, 0)$ where $k \neq 8$. Find k.

Solution. Denote the origin of the xy plane as O. Now denote the points $(0, 12), (0, -20), (8, 0), (k, 0)$ as A, B, C, and D respectively. Then by Power of a Point $AO \cdot BO = CO \cdot DO$. Plugging in values, we get that $DO = 30$. Therefore, $k = -30$.

7. Let pentagon $ABCDE$ be inscribed in circle O. Given that $\angle ABC = \angle CDE = 170°$, find $\angle AOE$.

Solution. Choose a point P on major arc $\widehat{AE}$ which doesn't contain C. Then $\angle APE = \angle APC + \angle EPC = 10° + 10°$ since $ABCP$ and $PCDE$ are cyclic. SO $\angle AOE = 2\angle APE = 40°$.

8. Let O denote the circumcenter of $\triangle ABC$. If $CA = 8$, $BC = 12$, and $AB = AO$, find the area of $\triangle ABC$.

Solution 1. Let K be the foot of the altitude from A to BC. Thus, it suffices to compute the value of AK. Let M be the midpoint of AC. Then $\angle OMA = \angle BKA = 90$. Also, $\angle AOC = \frac{1}{2} \cdot 2\angle B = \angle B$. Therefore, $\triangle ABK \sim \triangle AOM$. It follows that $AK = AM = 4$. So the area of $\triangle ABC$ is 24.

An important consequence of $\triangle ABK \sim AOM$ is that the orthocenter and circumcenter of a $\triangle ABC$ are *isogonal conjugates*. This means that if H and O denote the orthocenter and circumcenter respectively, then $\angle BAH = \angle CAO$.

Solution 2. Call the circumcenter O. Note that $\triangle ABO$ is equilateral. Therefore, $\angle AOB = 60$. Thus, minor arc $AB = 60$ so $\angle ACB = 30$. The area of the triangle is $\frac{1}{2} \cdot CA \cdot BC \sin 30° = 24$ by the sin area formula.

9. Let A, B, and C be 3 collinear points such that $AB = 2$ and $BC = 4$. Let point B be between A and C. Define P to be a point in the plane such that $\angle BPC = 30°$. Find the minimum possible length of AP.

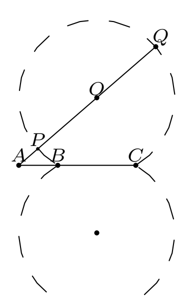
Solution. The key to solving this problem is determining the locus of all possible points P. We observe that the angle condition implies that P must lie on two arcs as shown in the figure below. Without loss of generality, we can look only at the "top" arc. Let the arc be a part of a circle with center O. Then $\angle BOC = 2\angle BPC = 60°$. So $\triangle BOC$ is equilateral, which means that the radius is 4. Now we are trying to compute AP. Seeing that we have the lengths AB and AC, we are motivated to apply power of a point. Let AP intersect the circle at Q. Then $AB \cdot AC = 12 = AP \cdot AQ$. We notice that as PQ increases, AP must decrease. Therefore, the minimum possible value of AP is achieved when PQ is maximal. This is when PQ is a diameter. Therefore, $PQ = 8$. So we have $AP \cdot (AP + 8) = 12$. So we must solve the quadratic

$$AP^2 + 8AP - 12 = 0.$$

Using the quadratic formula,

$$AP = \frac{-8 \pm \sqrt{64 + 48}}{2} = \frac{-8 \pm \sqrt{112}}{2} = \frac{-8 \pm 4\sqrt{7}}{2} = -4 \pm 2\sqrt{7}.$$

AP is clearly positive, so $AP = 2\sqrt{7} - 4$.



10. In $\triangle ABC$, let the angle bisector of $\angle ABC$ intersect the circumcircle at D. Let the perpendicular bisector of AB intersect BD at E. Given that $AB = 6$, $BC = 4$, and the area of $\triangle ABE$ is 3, find the length of DE.

    Solution. Notice that $\angle ADE = \angle C$ since they intercept the same arc of the circumcircle. Additionally, $\angle AED = \angle ABE + \angle BAE$ by exterial angle theorem. However, $\angle ABE = \frac{1}{2}\angle B$ since BD is an angle bisector. Also, since E lies on the perpendicular bisector of AD, then $\angle ABE = \angle BAE$. Therefore, $\angle AED = \angle B$. Now we know that $\triangle AED \sim \triangle ABC$. This means that

$$\frac{AB}{BC} = \frac{AE}{DE}.$$

    Now notice that the height of triangle has length 1 since the area of $\triangle ABE$ is 3. So by the Pythagorean Theorem $AE = \sqrt{1^2 + 3^2} = \sqrt{10}$. So

$$\frac{6}{4} = \frac{\sqrt{10}}{DE}.$$

    It follows that $DE = \frac{2\sqrt{10}}{3}$.