Amador Valley’s 6th Annual Geometry Bee
Invitational Round

May 23, 2013
Format

- Round 1: 10 problems, 2 minutes per problem
- Round 2: 10 problems, 2 minutes per problem
- Round 3: 10 problems, 3 minutes per problem
- Tiebreakers will be used to determine placements, if necessary
- Top 16 highest scores on Round 1 will advance to Round 2, top 8 highest scores on Round 2 will advance to Round 3
- All students that don’t advance above compete in the “Division 2” bracket, where they may win some other prizes, different from those that advance
Rules

- Write down your answers on the paper given.
- Unless otherwise stated, answers **must** be in simplest and exact form. Denominators must be rationalized, trigonometric functions must be simplified, common fractions must be used, etc.
- **No calculators!**
- Unless otherwise stated, assume angle measures are written in degrees.
- Good Luck!

<table>
<thead>
<tr>
<th>Incorrect</th>
<th>$1 \frac{1}{2} = 1.5$</th>
<th>$\cos(30^\circ)$</th>
<th>$\frac{1 + \sqrt{5}}{6 - 2\sqrt{5}}$</th>
<th>$\frac{\sqrt{5}}{2}$</th>
<th>$3^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct</td>
<td>$\frac{3}{2}$</td>
<td>$\frac{\sqrt{3}}{2}$</td>
<td>$\frac{2 + \sqrt{5}}{2} = 1 + \frac{\sqrt{5}}{2}$</td>
<td>$\frac{\sqrt{10}}{2}$</td>
<td>$243$</td>
</tr>
</tbody>
</table>
Round 1
10 problems, 2 minutes per problem
1. If $ABCD$ is a rectangle with area 24 and integer side lengths, find the difference between the maximum perimeter and the minimum perimeter.

**Answer** 30
2. In triangle $ABC$, $D$ is on $AC$ such that $\angle ABD = \angle CBD$. Given that $AB = 7$, $BC = 8$, and $AC = 5$, compute $\frac{AC}{AD}$.
3. In semicircle with center $O$, $AB$ is a diameter, and $K$ is on the circumference of the semicircle. Compute $AK^2 + BK^2$, given that $KO = 7$.

Answer: 196
4. In triangle $ABC$, given that $AB = 3$, $BC = 4$, and $AC = 5$, and $I$ as the incenter, compute $IA^2 + IB^2 + IC^2$. 

Answer: 17
5. Consider the 12-sided polygon $ABCDEFGHIJKL$, as shown. Each of its sides has length 8, and each two consecutive sides form a right angle. Suppose that $AG$ and $CH$ meet at $M$. What is the area of quadrilateral $ABCM$?

Answer: 352
6. Three mutually tangent spheres of radius 1 rest on a table. A sphere of radius 3 rests on top of them. What is the distance from the top of the larger sphere to the table?

Answer

$$4 + 2\sqrt{3}$$
7. A sphere is inscribed inside a cube. Given the cube’s side length as 6, compute the volume inside of the cube but outside of the sphere.

\[216 - 36\pi\]
8. Find the area of a regular hexagon inscribed in a circle inscribed in a regular triangle if the triangle has side length 4.

Answer: $2\sqrt{3}$
9. Brett, a dog with a poor sense of direction, begins 10 meters south of his doghouse. Suppose Brett begins to trot in a random direction until he is no longer moving towards the doghouse, that is, taking another step forward will increase the distance between Brett and his house. (Note that this could be right away: for example, if he begins by heading south.) At this point, if Brett is not home, he gives up and goes to sleep. What is the area, in square meters, of the region Brett might travel upon?

\[
\text{Answer: } 25\pi
\]
10. Which of the following is a Pythagorean Triple? List the correct letter.

A) (100361, 134679, 167961)  
B) (53672, 323830, 328248)  
C) (26128, 160254, 162370)  
D) (26836, 161915, 164124)  

No calculators! :)

Answer: D (26836, 161915, 164124)
Round 2
10 problems, 2 minutes per problem
1. In rectangle $ABCD$, $DC = 4$ and $BC = 3$. Points $F$ and $G$ are on $AB$ with $AF = 1$ and $BG = 2$. What is the perimeter of triangle $DEC$?

Answer: $4 + 4\sqrt{10} + 3 + 4\sqrt{13}$
2. Four circles with radii 2 are each tangent to two sides of a square and externally tangent to a circle of radius 3, as shown. What is the area of the square?

Answer: $2 + 4\sqrt{2}$
3. Equilateral triangle $ABC$ has a side length of 6, and it is circumscribed by circle $O$. $D$ is a point on minor arc $BC$ such that $\angle BAD = \angle CAD$. Compute $BD$.
4. In the figure, $AB$ and $CD$ are two perpendicular diameters of circle $O$. Chord $DF$ intersects $AB$ at $E$ with $DE = 5$ and $EF = 3$. What is the area of circle $O$?

Answer: $20\pi$
5. Consider a circle, with point $A$ randomly placed on the circumference. Point $B$ is placed randomly and independently of $A$, also on the circumference. Given we place point $C$ on minor arc $AB$, what is the probability that $C$ is placed so that arc $AC$ is more than $\frac{1}{6}$ of the total circumference of the circle?

Answer

$\frac{4}{9}$
6. In triangle $ABC$ and $DEF$, $DE = 3AB$, $DF = 3AC$, and $EF = 3BC$. Given that the area of $DEF$ is 120 more than $ABC$, compute the area of $DEF$. 

Answer 135
7. The figure shown is the union of a circle and semicircles of diameters 3 and 5, all of whose centers are collinear. What is the ratio of the area of the shaded region to that of the unshaded region?

Answer: \( \frac{3}{5} \)
8. Let $ABCD$ be a rectangle, and let $E$ be on $AB$ such that $2AE = EB$ and let $F$ be on $CD$ such that $CF = FD$. Let the intersection of $AF$ and $DE$ be $P$. Compute the fraction of the area of $ABCD$ that $BCFPE$ occupies.
9. Three spheres are externally tangent to each other and tangent to a common plane. If the spheres have radii 4, 9, and 16, what is the perimeter of the triangle with vertices at the intersection of each sphere with the plane?

Answer: 52
10. In the 2D world of Flatland on the XY plane, there is a single point light source and the ground is defined by $y = 0$. A 6 foot pole on $x = 0$ casts an 8 foot shadow left and an 8 foot pole on $x = 10$ casts a 6 foot shadow left. Find the height of the light source.

Answer
Round 2 Tiebreakers
You will have 3 minutes to solve one problem. When you solve a problem, raise your hand, and your proctor will grade it as well as assign you a time if you are correct. The participant with the lowest time moves on to the next round. In the case of both participants getting the problem wrong, we’ll move on to a second problem, and so forth.

» Go to Round 3
1. Ben is in a circular pen. If he goes 6 feet east, 4 feet north, or 8 feet west, he will encounter an electric fence and get shocked. What is the area, in square feet, of Ben’s pen?
2. A circle has radius 2 and is circumscribed about equilateral triangle $ABC$. If $X$ is the midpoint of $AC$ and $Y$ is on arc $AC$ such that $\angle YXA$ is right, then find $XY$. 

Answer
3. Monster Pac-Man is a growing sector of a circle. A baby Pac-Man starts with a radius of 0.25 cm and mouth opening of 11.25°. It doubles its radius and mouth opening each time it gobbles up a prey and continues growing until it reaches its adult size, which has a radius of at least 1.5 cm. When the baby Pac-Man stops growing (reaches adult size), what is its perimeter in cm?

Answer

Go to Round 3
4. Pac-Man has a 3-D cousin Pac-Ball, which is a sphere with a small wedge removed, as shown in the figure. If the radius \( r \) is 1 cm, and the opening angle \( \theta \) is 36°, what is the Pac-Ball’s surface-area-to-volume ratio?
5. What is the area of a triangle with side lengths 7, 8, and 9?
6. Find the area of a rectangle which has a perimeter of 30 and its length is 3 more than its width.

Answer

Go to Round 3
Round 3
10 problems, 3 minutes per problem
1. Akshay hates Rick, so he wants to cut up Rick’s homework, which is a regular sheet of paper. However, Rick infused the paper with magic, so that the paper will only separate after seven complete cuts have been made. Assuming that he cannot bend the paper, what is the maximum number of pieces Akshay can cut Rick’s homework into?

Answer: 29
2. Let $ABC$ be a triangle, and let the angle bisector of $\angle BAC$ intersect the circumcircle of $ABC$ at $D$. The perpendicular from $D$ to $BC$ intersects $BC$ at $E$. Given that $AB = 10$, $AC = 9$, and $BC = 8$, find $BE$. 

Answer 4
In a unit cube with side length 1 unit, vertices \( A \), \( C \), and \( H \) are adjacent to vertex \( D \). Consider the plane containing the points \( A \), \( C \), and \( H \). What is the shortest distance from \( D \) to this plane?

Answer: \( \sqrt{3} \)
4. Equilateral hexagon $A_1$ has an area of $6\sqrt{3}$. If $A_2$ is constructed by connecting the midpoints of the sides of $A_1$, and all $A_n$ are constructed similarly until an infinite number of hexagons are constructed, compute the sum of the areas enclosed by the hexagons.

**Answer** $24\sqrt{3}$
5. Cyclic quadrilateral $ABCD$ has side lengths $AB = \sqrt{48}$, $BC = \sqrt{49}$, $CD = \sqrt{51}$, and $DA = \sqrt{52}$. Find the area of the circumcircle of $ABCD$. 

Answer $25\pi$
6. In triangle $ABC$, points $D$ and $E$ lie on $BC$ and $AB$ respectively, so that $BD = 2DC$ and $BE = 2AE$. $AD$ and $CE$ intersect at $F$. The area of triangle $ABC$ is 24. Find the area of quadrilateral $FEBD$. 

Answer 4
7. Let $s$ denote the side length of the largest equilateral triangle that is inscribed inside a square of side length 3. Find $s^2$. 

Answer

$72 - 36\sqrt{3}$
8. Circular arcs $AC$ and $BC$ have centers at $B$ and $A$, respectively, and the circle in the figure is tangent to arcs $AC$ and $BC$ and to line $AB$. The length of arc $BC$ is $8\pi$. What is the area of the circle?
9. Triangle $ABC$ has $AB = 15$, $BC = 13$, and $AC = 4$. $D$ and $E$ are on $AB$ and $AC$, respectively, such that $[DBC] = 8$ and $[EBC] = 15$. $DC$ and $EB$ intersect at $M$. Find $[ADME]$. Note that $[XYZ]$ denotes the area of polygon $XYZ$. 

![Diagram of triangle ABC with points D, E, and M labeled, and lengths AB = 15, BC = 13, AC = 4, and areas $[DBC] = 8$ and $[EBC] = 15$. The lines $DC$ and $EB$ intersect at point M.]

Answer 23
10. Suppose I start with a regular tetrahedron with side length 2. In each step, I add a smaller regular tetrahedron onto every face such that the midpoints of each face are three of the vertices of the smaller tetrahedron. If I continue forever, what is the volume of the polyhedron?

The volume of a regular tetrahedron with side length $s$ is $\frac{s^3\sqrt{2}}{12}$.

Bonus: What shape does this become?
Round 3 Tiebreakers
You will have 3 minutes to solve one problem. When you solve a problem, raise your hand, and your proctor will grade it as well as assign you a time if you are correct. The participant with the lowest time moves on to the next round. In the case of both participants getting the problem wrong, we’ll move on to a second problem, and so forth.
1. \(A\) and \(B\) are points on a circle with center \(O\) and radius 8. Let \(P\) be the center of a circle internally tangent to circle \(O\) and tangent to \(OA\) and \(OB\). Given that \(AB = 3\), find the radius of \(P\).
2. A circle is divided into two sectors, one with a central angle of 120°, the other 240°. Each is wrapped into a cone. What is the ratio of the volume of the larger cone to the smaller cone?
3. A $45^\circ$ angle and a $30^\circ$ angle overlap, as shown, creating quadrilateral $ABCD$. Find the difference between $\angle ABC$ and $\angle ADC$ in degrees.
4. In the figure, angle $E$ equals $20^\circ$, and arc $AB = BC = CD$, what is $\angle ACD$?
5. Find the area of a right triangle with hypotenuse 25 and base 7.
Thank You for Coming!
Leftovers
Extra
1. A standard track is a total of 400 m long with 2 100 m straightaways and 2 semicircle turns. What is the area, in $m^2$, of the region contained by a track?

Answer: $10000\pi + 15000\pi$
2. Two identical spinning squares with side length 4 cm are pegged to a board such that their centers are 14 cm apart. Both squares begin aligned with a diagonal along the line running through their centers. A loop of string is tightly wrapped around both squares and turns along with them without slipping, causing the squares to spin in unison. A red dot is marked on the string at one of the corners on the axis connecting the squares. What distance does the dot travel before returning to its starting point?

NEEDS DIAGRAM

Answer

\[2\pi + \frac{9\pi}{\sqrt{2}}\]
3. Lines \( l \) and \( m \) are walls lying parallel to each other. Lux stands on a point on line \( m \), looking at line \( l \) making a \( 15^\circ \) angle with line \( l \). Katarina stands on line \( l \) to the right of Lux’s vision, and she looks at line \( m \) making a \( 30^\circ \) angle with line \( m \), but she sees Lux. The distance is Katarina’s vision, given that line \( m \) is a solid wall and Katarina cannot see past it, is \( a \sin b \), where \( b \) is in degrees. Compute \( a + b \).
4. Two parallel chords $AB$ and $CD$ in a circle have lengths 12 and 16, and the distance between them is 6. A third parallel chord $EF$ is between $AB$ and $CD$. The perpendicular distance between $AB$ and $EF$ is 2. What is the length of $EF$?
5. The side lengths of triangle $ABC$ are $BC = 9$, $AC = 12$, and $AB = 15$. Lines 2 units away from the sides of triangle $ABC$ are drawn, outside triangle $ABC$. Find the area of the triangle determined by these lines.
6. As shown in the figure, triangle $ABC$ is divided into six triangles by lines drawn from the vertices through a common interior point. The areas of these triangles are as indicated. Find the area of triangle $ABC$. 

![Diagram of triangle ABC divided into six triangles with areas indicated: 40, 84, 35, 30.]

Answer: 315
7. An acute isosceles triangle $ABC$ is inscribed in a circle. Tangents to the circle are drawn at $B$ and $C$, meeting at point $D$ with $\angle ABC = \frac{2}{3} \angle BDC$. What is the radian measure of $\angle BAC$?

Answer $\frac{\pi}{5}$
8. Find the length of the inradius of a triangle with side lengths 7, 7, and 10.

Answer: $\frac{5\sqrt{6}}{6}$
9. A triangle has vertices $P = (-8, 5)$, $Q = (-15, -19)$, and $R = (1, -7)$. The equation of the bisector of $\angle P$ can be written in the form $ax + 2y + c = 0$. Find $a + c$. 

Answer 89
10. In rectangle $ABCD$, points $F$ and $G$ lie on $AB$ such that $AF = FG = GB$ and $E$ is the midpoint of $DC$. Also, $AC$ intersects $EF$ at $H$ and $EG$ at $J$. The area of rectangle $ABCD$ is 140. Find the area of triangle $AHF$. 

Answer $28$
11. In triangle \( ABC \), \( AC = 12 \), \( BC = 5 \), and \( AB = 13 \). The infinite sequence of points \( C_1, C_2, C_3, C_4, \ldots \) is generated as follows: \( C_1 \) is the foot of the altitude from \( C \) to side \( AB \), \( C_2 \) is the foot of the altitude from \( C_1 \) to side \( AC \), \( C_3 \) is the foot of the altitude from \( C_2 \) to side \( AB \), and so on. Calculate the sum 
\[ CC_1 + C_1 C_2 + C_2 C_3 + C_3 C_4 + \cdots. \]

Answer 65
12. A point $P$ that is chosen in the interior of triangle $ABC$ such that when the lines are drawn through $P$ parallel to the sides of triangle $ABC$, the resulting triangles $t_1$, $t_2$, and $t_3$ in the figure have areas 4, 9, and 49, respectively. Find the area of triangle $ABC$. 

Answer: 144
13. Consider a cube $ABCDEFGH$ with side length 16 and $AE$ as a space diagonal. At the center of each face on this cube, a square hole of side length 8 is drilled all the way to the opposite face. Brian the ant is located on vertex $A$, and can only travel along the surface of the cube. Compute the least distance Brian can travel to reach vertex $E$.

Answer $\sqrt{2}$
14. A triangle is strong if all side lengths are integer length and it has two side lengths of 2013. How many strong triangles exist?

Answer: 4025
15. Given that a regular polygon has at most 15 sides and the number of sides is randomly chose, what is the probability that this polygon will have integer values for angles, in degrees?

Answer
16. Let $ABC$ be a triangle such that $AB = 5$, $BC = 12$, and $CA = 13$. A laser starts at point $B$ and shoots toward $CA$, bounces off and hits $BC$, and finally bounces off and ends in corner $A$. Compute the length of the distance this laser traveled.

Answer $5\sqrt{1321}$